UNIT - I
METRIC SPACES

Introduction

A Metric Space is a set equipped with a distance function, also called a metric, which enables us to measure the distance between two elements in the set.

1.1 Definition andExamples

Definition: A Metric Space is a non empty set M together with a function$\boldsymbol{d} : \boldsymbol{M} \times \boldsymbol{M} \rightarrow \boldsymbol{R}$ satisfying the following conditions.

(i) $d(x, y) \geq 0$ for all $x, y \in M$

(ii) $d(x, y) = 0$ if and only if $x = y$

(iii) $d(x, y) = d(y, x)$ for all $x, y \in M$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$**[Triangle Inequality]**

d is called a **metric** or**distance function** on M and $\boldsymbol{d(x, y)}$is called the distance between x and y in M. The metric space M with the metric d is denoted by (M, d) or simply by M when the underlying metric is clear from the context.

Example 1.

Let $\boldsymbol{R}$ be the set of all real numbers. Define a function $d : M \times M \rightarrow R$by $d(x, y) | x - y |$. Then d is a metric on $\boldsymbol{R}$ called the usual metric on$\boldsymbol{R}$**.**

Proof.

Let $x, y \in \boldsymbol{R}$.

Clearly $(x, y) = | x - y | \geq 0$. Moreover,

$d(x, y) = 0 \quad \Leftrightarrow |x - y| = 0.$
$\Leftrightarrow x - y = 0.$
$\Leftrightarrow x = y$

$d(x, y) = | x - y|$

$= | y - x|$

$= d(y, x).$

$\therefore d(x, y) = d(y, x).$

Let $x , y , z \in \boldsymbol{R}$.

$d(x , z) = | x - z |$

$= | x - y + y - z |$

$\leq | x - y | + | y - z |$

$= d(x , y) + d(y , z)$.

$\therefore d(x , z) \leq d(x , y) + d(y , z)$.

Hence d is a metric on $\boldsymbol{R}$.

Note. When R is considered as a metric space without specifying its metric, it is the usual metric.

Example 2

Let $\boldsymbol{M}$ be any non-empty set. Define a function $\boldsymbol{d} : \boldsymbol{M} \times \boldsymbol{M} \to \boldsymbol{R}$ by

$\boldsymbol{d(x, y)} = \begin{cases} \boldsymbol{0 \; if \; x = y} \\ \boldsymbol{1 \; if \; x \neq y} \end{cases}$ Then $\boldsymbol{d}$ is a metric on $\boldsymbol{M}$ called thediscrete metric **or** trivial metric **on** $\boldsymbol{M}$.

Proof.

Let $x , y \in M$.

Clearly $d(x , y) \geq 0$and $d(x , y) = 0 \Leftrightarrow x = y$.

Also, $d(x , y) = \begin{cases} 0 \; if \; x = y \\ 1 \; if \; x \neq y \end{cases}$

$= d(y , x)$.

Let $x , y , z \in M$.

We shall prove that $d(x , z) \leq d(x , y) + d(y , z)$.

Case (i) Suppose $x = y = z$.

Then $(x , z) = 0 , d(x , y) = 0 , d(y , z) = 0$.

$\therefore d(x , z) \leq d(x , y) + d(y , z)$.

Case (ii) Suppose $x = y$ and z distinct.

Then $(x , z) = 1 , d(x , y) = 0 , d(y , z) = 1$.

$\therefore d(x , z) \leq d(x , y) + d(y , z)$.

Case (iii) Suppose $x = z$ and y distinct.

Then,

$d(x,z) = 0, d(x,y) = 1, d(y,z) = 1.$
$\therefore d(x,z) \leq d(x,y) + d(y,z).$

Case (iv) Suppose $y = z$ and x distinct.

Then,

$d(x,z) = 1, d(x,y) = 1, d(y,z) = 0.$
$\therefore d(x,z) \leq d(x,y) + d(y,z).$

Case (v) Suppose $x \neq y \neq z$.

Then $d(x,z) = 1, d(x,y) = 1, d(y,z) = 1.$

$\therefore d(x,z) \leq d(x,y) + d(y,z).$

In all the cases,

$d(x,z) \leq d(x,y) + d(y,z).$
Hence d is a metric on M.

1.2.Open Sets in aMetricSpace

Definition: Let (M,d) be a metric space. Let $a \in M$ and r be a positive real number. The open ball or the open sphere with center a and radius$r$ is denoted by$B_d(a,r)$ and is the subset of M defined by $B_d(a,r) = \{x \in M \,/\, d(a,x) < r\}$. We write $B(a,r)$ for $B_d(a,r)$ if the metric d under consideration isclear.

Note. Since $(a,a) = 0 < r, a \in B_d(a,r)$.

Examples:

1. In $\boldsymbol{R}$with usual metric $B(a,r) = (a - r, a + r)$.
2. In $\boldsymbol{R}^2$with usual metric $B(a,r)$ is the interior of the circle with center $a$and radius$r$.

Definition: Let (M,d) be a metric space. A subset A of M is said to be open in M if for each $x \in A$ there exists a real number $r > 0$ such that $B(x,r) \subseteq A$.

Note. By the definition of open set, it is clear that ϕ and M are open sets.

Examples:

1. Any open interval $(a\,,b)$ is an open set in $\boldsymbol{R}$ with usual metric.

For,
Let $x\ \in\ (a\,,b)$.
Choose a real number r such that $0\ <\ r\ \leq\ min\{\,x-a\,,b-x\,\}$.
Then $B(x\,,r)\ \subseteq\ (a\,,b)$.
$\therefore\ (a\,,b)$is open in R.
1. Every subset of a discrete metric space M isopen. For,
Let A be a subset of M.
If $A\ =\ \phi$, then A is open.
Otherwise, let $x\ \in\ A$.
Choose a real number r such that $0\ <\ r\ \leq\ 1$. Then
$B(x\,,r)\ =\ \{\,x\,\}\ \subseteq\ A$ and hence A is open.
2. Set of all rational numbers $\boldsymbol{Q}$ is not open in $\boldsymbol{R}$. For,
Let $x \in \boldsymbol{Q}$.
For any real number $r\ >\ 0, B(x\,,r)\ =\ (x\ -\ r\,,x\ +\ r)$ contains both rational and irrational numbers.
$\therefore\ B(x\,,r)\ \nsubseteq\ \boldsymbol{Q}$and hence $\boldsymbol{Q}$ is not open.

Theorem 1.1
Let $(M\,,d)$ be a metric space. Then each open ball in M is an open set.

Proof.

Let $B(a\,,r)$ be an open ball in M.
Let $x\ \in\ B(a\,,r)$.
Then $d(a\,,x)\ <\ r$.

Take$r_1\ =\ r - d(a\,,x)$.Then $r_1 >\ 0$.

We claim that$B(\,x\,,r_1)\ \subseteq B(\,a\,,r)$.

Let $y \in B(\,x\,,r_1)$.

Then $(x\,,y)\ < r_1$.

Now,

$d(a,y) \leq d(a,x) + d(x,y)$
$< d(a,x) + r_1$

$= d(a,x) + r - d(a,x) = r.$

$\therefore d(a,y) < r.$

$\therefore y \in B(a,r).$

$\therefore B(x,r1) \subseteq B(a,r).$

Hence $B(a,r)$ is an open ball.

Theorem1.2

In any metric space M, the union of open sets is open.

Proof.

Let $(\boldsymbol{M}, \boldsymbol{d})$ be a Metric Space.

Let$\{A_i / i \in I\}$ a family of open sets in M.

We have to prove $A = \cup A_i$ is open in M.

If $A = \phi$ then A is open.

∴Let $A \neq \phi$. Let $x \in A$.

Then $x \in A_i$ for some $\in I$.

Since A_i is open, there exists an open ball $B(x,r)$ such that $B(x,r) \subseteq A_i$.

$\therefore B(x,r) \subseteq A.$
Hence A is open in M.

Theorem 1.3

In any metric space M, the intersection of a finite number of open sets is open.

Proof:

Let $A_1, A_2, \ldots, A_n$ be open sets in M.

We have to prove $A = A_1 \cap A_2 \cap \ldots \cap A_n$is open in M.

If $A = \phi$ then A is open.

∴Let $A \neq \phi$. Let $x \in A$.
Then, $x \in A_i$ for each $i = 1, 2, \dots, n$.

Since each A_i is open, there exists an open ball $B(x, r_i)$ such that $B(x, r_i) \subseteq A_i$.

Take $r = min\{r_1, r_2, \dots, r_n\}$.
Clearly, $r > 0$ and
$B(x, r) \subseteq B(x, r_i)$ for all $i = 1, 2, \dots, n$.

Hence $B(x, r) \subseteq A_i$ for each $i = 1, 2, \dots, n$.

∴ $B(x, r) \subseteq A$.

∴ A is open in M.

Theorem 1.4

Let (M, d) be a metric space and $A \subseteq M$. Then A is open in M if and only if A can be expressed as union of open balls.

Proof : Suppose that A is open in M.

Then for each $x \in A$ there exists an open ball $B(x, r_x)$ such that, $B(x, r_x) \subseteq A$.

$A = \cup_{x \in A} B(x, r_x)$.

Thus A is expressed as union of open balls.

Conversely, assume that A can be expressed as union of open balls. Since open balls are open and union of open sets is open, A is open.

1.2 Interior of aset

Definition Let (M, d) be a metric space and $A \subseteq M$. A point $x \in A$ is said to be an interior point of A if there exists a real number $r > 0$ such that $B(x, r) \subseteq A$. The set of all interior points is called as interior of A and is denoted by $\boldsymbol{Int}\, A$.

Note: $\boldsymbol{Int}\, A \subseteq A$.

Example: In $\boldsymbol{R}$ with usual metric, let $A = [1, 2]$. 1 is not an interior points of A, since for any real number > 0, $B(1, r) = (1 - r, 1 + r)$ contains real numbers less than 1. Similarly, 2 is also not an interior point of A. In fact every point of $(1, 2)$ is a limit point of A. Hence $\boldsymbol{Int} A =$

$(1,2)$.

Note:
(1)$\textbf{Int}\,\boldsymbol{\phi} = \boldsymbol{\phi}$ *and* $\textbf{Int}\,M = M$.
(2)A *is open* $\Leftrightarrow \textbf{Int}\,A = A$.
(3)$A \subseteq B \Rightarrow \textbf{Int}\,A \subseteq \textbf{Int}B$.

Theorem1.5

Let (M, d) be a metric space and $A \subseteq M$. Then $\textbf{Int}\,A$ = Union of all open sets contained in A.

Proof.

Let G = ∪{B/B is an open set contained in A}

we have to prove $\textbf{Int}\,A = G$.
Let $x \in \textbf{Int}\,A$.

Then x is an interior point of A.

∴there exists a real number $r > 0$ such that $B(x, r) \subseteq A$.

Since open balls are open, $B(x, r)$ is an open set contained in A.

∴ $B(x, r) \subseteq G$.
∴ $x \in G$.

∴ $\textbf{Int}\,A \subseteq G$ .... (1)

Let ∈ G .

Then there exists an open set$B$ such that $B \subseteq A$and $x \in B$.

Since B is open and $x \in B$, there exists a real number $r > 0$ such that $B(x, r) \subseteq B \subseteq A$.

∴ $x$is an interior point of A.

∴ $x \in \textbf{Int}\,A$.

∴ $G \subseteq \textbf{Int}A$.. (2)

From (1) and (2), we get $\textbf{Int}\,A = G$.
Note:$\textbf{Int}\,A$ is an open set and it is the largest open set contained in A.

Theorem1.6

Let M be a metric space and $A , B \subseteq M$. Then

(1) $\boldsymbol{Int}\ (A \cap B) = (\boldsymbol{Int}\ A) \cap (\boldsymbol{Int} A)$

(2) $\boldsymbol{Int}\ (A \cup B) \supseteq (\boldsymbol{Int}\ A) \cup (\boldsymbol{Int} A)$

Proof.

(1)$A \cap B \subseteq A \Rightarrow \boldsymbol{Int}(A \cap B) \subseteq \boldsymbol{Int}\ A$.

Similarly, $\boldsymbol{Int}\ (A \cap B) \subseteq \boldsymbol{Int}\ B$.
$\therefore \boldsymbol{Int}\ (A \cap B) \subseteq (\boldsymbol{Int}\ A) \cap (\boldsymbol{Int} A)$ .... (a)

$\boldsymbol{Int} A \subseteq A$ and $\boldsymbol{Int}\ B \subseteq B$.

$\therefore$ $(\boldsymbol{Int}\ A) \cap (\boldsymbol{Int}\ A) \subseteq A \cap B$

Now, $(\boldsymbol{Int}\ A) \cap (\boldsymbol{Int}\ A)$ is an open set contained in $\cap B$.

But, $\boldsymbol{Int}\ (A \cap B)$ is the largest open set contained in $\cap B$.
$\therefore (\boldsymbol{Int}\ A) \cap (\boldsymbol{Int}\ A) \subseteq \boldsymbol{Int}\ (A \cap B)$ .... (b)

From (a) and (b) , we get $\boldsymbol{Int}(A \cap B) = (\boldsymbol{Int} A) \cap (\boldsymbol{Int} A)$

$A \subseteq A \cup B \Rightarrow \boldsymbol{Int} A \subseteq \boldsymbol{Int}(A \cup B)$
Similarly, $\boldsymbol{Int}\ \boldsymbol{B} \subseteq \boldsymbol{Int}\ (A \cup B)$
$\therefore \boldsymbol{Int}(A \cup B) \supseteq (\boldsymbol{Int} A) \cup (\boldsymbol{Int} A)$

Note1.7: $\boldsymbol{Int}(A \cup B)$need not be equal to$(\boldsymbol{Int} A) \cup (\boldsymbol{Int}\ A)$
For,
In$\boldsymbol{R}$with usual metric,
let$A = (0,1]$ and $B = (1,2). A \cup B = (0,2)$.
$\therefore \boldsymbol{Int}(A \cup B) = (0,2)$
Now, $\boldsymbol{Int} A(0,1)$ and $\boldsymbol{Int} B = (1,2)$ and hence $(\boldsymbol{Int} A) \cup (\boldsymbol{Int} A) = (0,2) – \{2\}$.
$\therefore \boldsymbol{Int}(A \cup B) \neq (\boldsymbol{Int} A) \cup (\boldsymbol{Int}\ A)$

1.2.Subspace

Definition:

Let(M , d) be a metric space. Let M_1 be a nonempty subset of M. Then M_1 is also a metric space under the same metric d. We call (M_1, d) is a subspace of (M, d).

Theorem1.8 Let M be a metric space and M_1 a subspace of M. Let $A \subseteq M_1$. Then A_1 is open in M_1 if and only if A_1 = A ∩ M_1 where A is open in M.

Proof:

Let $\boldsymbol{M_1}$ be a subspace of $\boldsymbol{M}$. Let $\boldsymbol{a} \in \boldsymbol{M_1}$.

Let $M_1(a\ ,r)$ be the open ball in M_1 with center a and radius r.

Then $B_1(a,r) = B(a,r) \cap M_1$where $B(a,r)$ is the open ball in M with center a and radius r.

Then $B_1(a,r) = \{x \in M_1/d(a,x) < r\}$.

Also, $B(a,r) = \{x \in M/d(a,x) < r\}$.

Hence,$B_1(a,r) = B(a,r) \cap M_1$.

Let A_1 be an open set in M_1.

Then,

$$A_1 = B_1(x,r(x))$$

$= \bigcup_{x\in A_1}[B(x,r(x)) \cap M_1]$

$=\left[\bigcup_{x\in A_1} B(x,r(x))\right] \cap M_1$

= A $\cap M_1$

Where A = $\bigcup_{x\in A_1} B(x,r(x))$which is open in M.

Conversely, let $A = G \cap M_1$ where G is open in M.
We shall prove that A_1 is open inM.
Let $x \in A_1$.
Then $x \in A\ and\ x \in M_1$.
Since A is open in M, there exists an open ball B(x,r) such that B(x,r)⊆A.
$\therefore B(x,r)M_1 \cap\subseteq A \cap M_1$.

i.e. $B_1(x,r) \subseteq M_1$.

$\therefore A_1$is open in M_1.

1.2.Bounded Sets in a Metric space.

Definition: Let(M,d)be a metric space. A subset A of M is said to be bounded if there exists a positive real number k such that $d(x,y) \leq k \ \forall\ x,y \in A$.

Example: Any finite subset A of a metric space $(M\ ,d)$ is bounded. For,
Let A be any finite subset of M.

If $A = \phi$ then A is obviously bounded.

Example:$[0,1]$ is a bounded subset of $\boldsymbol{R}$ with usual metric since $d(x, y) \leq 1$ for all $x, y \in [0,1]$.

Example:$(0, \infty)$ is an unbounded subset of $\boldsymbol{R}$.

Example: Any subset A of a discrete metric space M is bounded since

$d(x, y) \leq 1\ for\ all\ x, y \in A.$

Note: Every open ball $B(x, r)$ in a metric space (M, d) is bounded.

Definition: Let(M, d) be a metric space and $A \subseteq M$. The diameter of A, denoted by $d(A)$, is defined by $d(A) = l.u.b\ \{d(x, y)/x, y \in A\}$.

Example: InR with usual metric the diameter of any interval is equal to the length of the interval. The diameter of [0,1] is 1.

UNIT – II
CLOSED SETS

2.1.ClosedSets

Definition: A subset A of a metric space M is said to be closed in M if its complement is open in M.

Examples

1. In $\boldsymbol{R}$with usual metric any closed interval $[a, b]$ isclosed. For,

$[a\ ,b]^c = \boldsymbol{R} - [a\ ,b] = (\ -\infty\ ,a) \cup (b\ ,\infty)$.

$(-\infty, a)$and(b, ∞) are open sets in R and hence $(-\infty\ ,a)\ \cup (b\ ,\infty)$ is open in **R**.

i.e. $[a\ ,b]^c$ is open in $\boldsymbol{R}$.

$\therefore$ $[a\ ,b]$is open in $\boldsymbol{R}$.

1. Any subset A of a discrete metric space M is closed since $A^c$is open as every subset of M isopen.

Note. In any metric space M, ϕ and M are closed sets since $\phi^c = M$ and $M^c = \phi$ which are open in M. Thus ϕ and M are both open and closed in M.

Theorem 2.1.

In any metric space M, the union of a finite number of closed sets is closed.

Proof:

Let $(\boldsymbol{M}, \boldsymbol{d})$be a Metric space.

Let $\boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]$ be a closed ball in $\boldsymbol{M}$.

Case (i) Suppose $\boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]^c = \boldsymbol{\phi}$

$\therefore \boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]^c$is open and hence $\boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]$ is closed.

Case (ii) Suppose $\boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]^c \neq \boldsymbol{\phi}$

Let $\boldsymbol{x} \in \boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]^c$.

$\therefore \boldsymbol{x} \notin \boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]^c$.

$\therefore \boldsymbol{d}(\boldsymbol{a}, \boldsymbol{x}) > \boldsymbol{r}$

$\therefore \boldsymbol{d}(\boldsymbol{a}, \boldsymbol{x}) - \boldsymbol{r} > \boldsymbol{0}$.

Let $\boldsymbol{r_1} = \boldsymbol{d}(\boldsymbol{a}, \boldsymbol{x}) - \boldsymbol{r}$.

We claim that $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{r_1}) \subseteq \boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]^{\boldsymbol{c}}$.

Let $\boldsymbol{y} \in \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{r_1})$.

Then $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) < \boldsymbol{r_1} = \boldsymbol{d}(\boldsymbol{a}, \boldsymbol{x}) - \boldsymbol{r}$.

$\therefore \boldsymbol{d}(\boldsymbol{a}, \boldsymbol{x}) > \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{r}$.

Now, $\boldsymbol{d}(\boldsymbol{a}, \boldsymbol{x}) \leq \boldsymbol{d}(\boldsymbol{a}, \boldsymbol{y}) + \boldsymbol{d}(\boldsymbol{y}, \boldsymbol{x})$.

$\boldsymbol{d}(\boldsymbol{a}, \boldsymbol{y}) \geq \boldsymbol{d}(\boldsymbol{a}, \boldsymbol{x}) - \boldsymbol{d}(\boldsymbol{y}, \boldsymbol{x})$.

$> \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{r} - \boldsymbol{d}(\boldsymbol{y}, \boldsymbol{x})$.

$= \boldsymbol{r}$.

Thus , $\boldsymbol{d}(\boldsymbol{a}, \boldsymbol{y}) > \boldsymbol{r}$.

$\therefore \boldsymbol{y} \notin \boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]$.

Hence $\boldsymbol{y} \in \boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]^{\boldsymbol{c}}$.

$\therefore \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{r_1}) \subseteq \boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]^{\boldsymbol{c}}$.

$\therefore \boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]^{\boldsymbol{c}}$is open in $\boldsymbol{M}$.

$\therefore$ $\boldsymbol{B}[\boldsymbol{a}, \boldsymbol{r}]$is closed in $\boldsymbol{M}$.

Theorem 2.2

In any metric space M, arbitary intersection of closed sets is closed.

Proof:

Let $(\boldsymbol{M}, \boldsymbol{d})$ be a metric space.

Let$\{\boldsymbol{A_i} / \boldsymbol{i} \in \boldsymbol{I}\}$be a family of closed sets in $\boldsymbol{M}$.

We have to prove $\bigcap_{i \in I} A_i$ is closed.

Wehave$(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} {A_i}^c$
(byDeMorgan's law)

Since A_i is closed ${A_i}^c$ is open.

Hence $\bigcup_{i \in I} {A_i}^c$isopen.
$\therefore$ $(\bigcap_{i \in I} A_i)^c$is open in M.

$\therefore \bigcap_{i \in I} A_i$ is closed in M.

Theorem 2.3

Let M_1 be a subspace of a metric space M. Let $F_1 \subseteq M_1$. Then F_1 is closed in M_1 if and only if $F_1 = F \cap M_1$ where F is a closed set in M.

Proof.

Suppose that $F$is closed in M_1.
Then $M_1 - F_1$ is open inM_1.
$\therefore M_1 - F_1 = A^c \cap M_1$where$A$ is open in M.

Now, $F_1 = A \cap M_1$.

Since A is open in M, A^c is closed in M.

Thus, $F_1 = F \cap M_1$where $F = A^c$ is closed in M.

Conversely, assume that $F_1 = F \cap M_1$ where F is closed in M.

Since F is closed in M, $F^c$is open in M.

$\therefore F^c \cap M_1$ is open in M_1.

Now, $M_1 - F_1 = F^c \cap M_1$ which is open inM_1.

$\therefore F_1$is closed inM_1.

Proof of the converse is similar.

2.1.Closure.

Definition: Let$A$ be a subset of a metric space (M, d). The closure of A, denoted by $\bar{A}$ is defined to be the intersection of all closed sets which contain A.

i.e. $\bar{A} = \cap \{B / B \text{ is closed in } M \text{ and } A \subseteq B\}$.

Note

(1) Since intersection of closed sets is closed, $\bar{A}$ is closedset.

(2) $A_1$is the smallest closed set containing$A$.

(3) A is closed $\Leftrightarrow$ A =$\bar{A}$.

Theorem 2.4:
Let (M, d) be a metric space. Let $A, B \subseteq M$. Then

(i) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

(ii)$\overline{A \cup B} = \bar{A} \cup \bar{B}$

(iii)$\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Proof:

(i)let$\boldsymbol{A} \subseteq \boldsymbol{B},$

Now, $\bar{B} \supseteq B \supseteq A.$
Thus $\bar{B}$is a closed set containing $A.$

But $\bar{A}$ is the smallest closed set containing $A.$

$\therefore \bar{A} \subseteq \bar{B}.$

(ii)we have A ⊆ A ∪B.

$\therefore \bar{A} \subseteq \overline{A \cup B}.$ (by (1)).

Similarly ,$\bar{B} \subseteq \overline{A \cup B}.$

$\therefore \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$----------------------(1)

Now,$\bar{A}$ is a closed set containing A and $\bar{B}$ is a closed set containing B.

$\therefore \bar{A} \cup \bar{B}$ is a closed set containing $A \cup B.$

But $\overline{A \cup B}$ is the smallest closed set containing $A \cup B.$

$\therefore \overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$----------------------(2)

From (1) and (2) we get $\overline{A \cup B} = \bar{A} \cup \bar{B}.$

(iii)we have $A \cap B \subseteq A$

$\overline{A \cap B} \subseteq \bar{A}$ (by(i)).

Similarly,$\overline{A \cap B} \subseteq \bar{B}$

$\therefore \overline{A \cap B} \subseteq \bar{A} \cap \bar{B}.$

Note:$\overline{A \cap B}$ need not be equal to $\bar{A} \cap \bar{B}.$

2.1.Limit Point

Definition: Let (M, d)be a Metric space.Let $A \subseteq M$.Let $x \in M$. Then x is called a limit point of A if every open ball with centre $\boldsymbol{x}$ contains atleast one point of A differ from x.
$(i.e)B(x, r) \cap (A - \{x\}) \neq \phi\ for\ all\ r > 0$. The set of all limit points of A is called the **derived set** of A and is denoted by $D(A)$

Theorem 2.4
Let (M, d) be a metric space and $A \subseteq M$. Then x is a limit point of A if and only if every open ball with center x contains infinite number of points of A.

Proof:

Let $\boldsymbol{x}$ be a limit point of $\boldsymbol{A}$.

Suppose an open ball $\boldsymbol{B(x,r)}$ contains only a finite number of points of $\boldsymbol{A}$.

$\boldsymbol{B(x,r) \cap (A-\{x\}) = \{x_1, x_2, \ldots\ldots, x_n\}}$

let $\boldsymbol{r_1 = min\{d(x,x_i)/i = 1, 2, \ldots\ldots, n\}}$.

Since $\boldsymbol{x \neq x_i, d(x,x_i) > 0\ for\ all\ i = 1, 2, \ldots\ldots, n}$ and hence $\boldsymbol{r_1 > 0}$.

Also $\boldsymbol{B(x,r) \cap (A-\{x\}) = \phi}$.

$\therefore$ $\boldsymbol{x}$is not a limit point of A which is a contradiction. Hence every ball with center $\boldsymbol{x}$ contains infinite number of points of $\boldsymbol{A}$.

The converse is obvious.
Corollary 1: Any finite subset of a metric space has no limit points.

Theorem 2.5

Let $\boldsymbol{M}$ be a metric space and $\boldsymbol{A \subseteq M}$. then $\boldsymbol{\overline{A} = A \cup D(A)}$.

Proof: Let $\boldsymbol{x \in A \cup D(A)}$.we shall prove that $\boldsymbol{x \in \overline{A}}$.

Suppose$\boldsymbol{x \notin \overline{A}}$.

$\therefore$ $\boldsymbol{x \in M - \overline{A}}$and since $\boldsymbol{\overline{A}}$ is closed $\boldsymbol{M - \overline{A}}$ is open.

$\therefore$ Thereexists an open ball $\boldsymbol{B(x,r) \subseteq M - \overline{A}}$.

$\therefore$ $\boldsymbol{B(x,r) \cap \overline{A} = \phi}$.

$\therefore$ $\boldsymbol{B(x,r) \cap A = \phi}$. (since $\boldsymbol{A \subseteq \overline{A}}$).

$\boldsymbol{x \notin A \cup D(A)}$which is a contradiction.

$\therefore$ $\boldsymbol{x \in \overline{A}}$.

$\therefore$ $\boldsymbol{A \cup D(A) \subseteq \overline{A}}$. Now let $\boldsymbol{x \in \overline{A}}$. To prove $\boldsymbol{x \in A \cup D(A)}$.

If $\boldsymbol{x \in A}$clearly $\boldsymbol{x \in A \cup D(A)}$.

Suppose $\boldsymbol{x \notin A}$. We claim that $\boldsymbol{x \in D(A)}$.

Suppose $\boldsymbol{x \notin D(A)}$. Then there exists an open ball $\boldsymbol{B(x,r)}$ such that $\boldsymbol{B(x,r) \cap A = \phi}$.

$\therefore$ $\boldsymbol{B(x,r)^c \supseteq A}and\boldsymbol{B(x,r)^c}$ is closed.

But $\overline{A}$ is the smallest closed set containing A.

$\therefore \overline{A} \subseteq B(x,r)^c$.

but $x \in \overline{A}$ and $x \notin B(x,r)^c$ which is a contradiction.

Hence $x \in D(A)$.

$\therefore x \in A \cup D(A)$.

$\therefore \overline{A} \subseteq A \cup D(A)$

Hence $\therefore A \cup D(A) = \overline{A}$.

Corollary 1: A is closed iff A contains all its limit points.

(i.e.) A is closed iff $D(A) \subseteq A$.

Proof: A is closed $\Leftrightarrow A = \overline{A}$ (by theorem 2.13)

$\Leftrightarrow A = A \cup D(A)$.

$\Leftrightarrow D(A) \subseteq A$.

Corollary 2: $x \in \overline{A} \Leftrightarrow B(x,r) \cap A \neq \phi$ for all $r > 0$.

Proof: let $x \in \overline{A}$, then $x \in A \cup D(A)$.

$\therefore x \in A$ or $x \in D(A)$.

If $x \in A$ then $x \in B(x,r) \cap A$.

if $x \in D(A)$ then $B(x,r) \cap A \neq \phi$ for all $r > 0$.

Hence in both cases $B(x,r) \cap A \neq \phi$ for all $r > 0$.

Conversely Suppose $B(x,r) \cap A \neq \phi$ for all $r > 0$.

We have to prove that, $x \in \overline{A}$.

If $x \in A$ trivially $x \in \overline{A}$.

Let $x \notin A$. Then $A - \{x\} = A$.

$\therefore B(x,r) \cap A - \{x\} \neq \phi$.

$\therefore x \in D(A)$.

$\therefore x \in \overline{A}$.

Corollary 3:

$x \in \overline{A} \Leftrightarrow G \cap A \neq \phi$ for every open set Gcontaining x.

Proof: Let $x \in \overline{A}$.

Let G be an open set containing x.then there exists $r > 0$ such that $B(x,r) \subseteq G$.

Also, since $x \in \overline{A}$, $B(x,r) \cap A \neq \phi$**.**

$\therefore$ $G \cap A \neq \phi$.

Conversely suppose $G \cap A \neq \phi$ for every open set Gcontaining x.

Since $B(x,r)$is an open set containing x,we have $B(x,r) \cap A \neq \phi$**.**
$\therefore$ $x \in \overline{A}$.

2.1.Dense sets

Definition:A subset A of a metric space M is said to be dense in Mor every where dense if $\overline{A} = M$.

Definition: A metric space M is said to be separable if there exists a countable dense subset in M.

Note :

(1) Any countable metric space is separable.
(2)Any uncountable discrete metric space is not separable.

Theorem 2.6:

Let M be a metric space and $A \subseteq M$.then the following are equivalent.

(i) $A$is dense in M.

(ii) The only closed set which contains $A$is M.

(iii) The only open set disjoint from $A$is ϕ.

(iv) A intersects every non empty open set.

(v) Aintersects every open ball.

Proof:

(i)⇒(ii).

Suppose $\boldsymbol{A}$is dence in $\boldsymbol{M}$.

Then $\overline{\boldsymbol{A}} = \boldsymbol{M}.$ -------------------------- (1)

Now, let $\boldsymbol{F} \subseteq \boldsymbol{M}$be closed set containing $\boldsymbol{A}.$

Since $\overline{\boldsymbol{A}}$ is a closed set containing $\boldsymbol{A}$,we have $\overline{\boldsymbol{A}} \subseteq \boldsymbol{F}$.

Hence $\boldsymbol{M} \subseteq \boldsymbol{F}$.(by (1))

$\therefore \boldsymbol{M} = \boldsymbol{F}$.

Hence,The only closed set which contains $\boldsymbol{A}$ is $\boldsymbol{M}$.

(ii)⇒(iii)

Suppose (iii) is not true.

Then there exists a non emptyopenset$\boldsymbol{B}$ such that,$\boldsymbol{B} \cap \boldsymbol{A} = \boldsymbol{\phi}$.

$\therefore \boldsymbol{B}^c$is z closed set and $\boldsymbol{B}^c \supseteq \boldsymbol{A}$.

Further, since $\boldsymbol{B} \neq \boldsymbol{\phi}$ we have $\boldsymbol{B}^c \neq \boldsymbol{M}$ which is a contradiction to (ii).

Hence (ii))⇒(iii).

Obviously, (iii)⇒(iv).

(iv)⇒(v),since every open ball is an openset.

(iv) ⇒(i)

Let $\boldsymbol{x} \in \boldsymbol{M}$. Suppose every open ball $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{r})$intersects $\boldsymbol{A}$.

Then by corollary, $\boldsymbol{x} \in \overline{\boldsymbol{A}}$.

$\therefore \boldsymbol{M} \subseteq \overline{\boldsymbol{A}}$. But trivially $\overline{\boldsymbol{A}} \subseteq \boldsymbol{M}$.

$\therefore \overline{\boldsymbol{A}} = \boldsymbol{M}$.

$\therefore \boldsymbol{A}$is dense in $\boldsymbol{M}$.

2.2.Completeness

Definition: let $(\boldsymbol{M}, \boldsymbol{d})$ ba a metric space. Let $(\boldsymbol{x_n} = \boldsymbol{x_1}, \boldsymbol{x_2}, \dots., \boldsymbol{x_n} \dots)$ be a sequence of points in $\boldsymbol{M}$. Let $\boldsymbol{x} \in \boldsymbol{M}$. We say that $(\boldsymbol{x_n})$ converges to $\boldsymbol{x}$ if given $\boldsymbol{\varepsilon}$ >0 there exists a positive integer $\boldsymbol{n_0}$

such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$.Also x is called a limit of (x_n).

If (x_n) converges to x we write $\lim_{n\to\infty} x_n = x$or $(x_n) \to x$.

Note 1:$(x_n) \to x$ iff for each open ball $B(x, \varepsilon)$ with cebtre x there exists a positive integer n_0 such that $x_n \in B(x, \varepsilon)$ for all $n \geq n_0$.

Thus the open ball $B(x, \varepsilon)$ contains all but a finite number of terms of the sequence.

Note 2:$(x_n) \to x$ iff the sequence of real numbers $d\big((x_n, x)\big) \to 0$.

Theorem2.6:

For a convergent sequence (x_n) the limit is unique.

Proof: Suppose $(x_n) \to x$and $(x_n) \to y$.

Let $\varepsilon > 0$ be given. Then there exist positive integers $n_1$and n_2 such that

$d(x_n, x) < \frac{1}{2}\varepsilon$for all $n \geq n_1$and $d(x_n, y) < \frac{1}{2}\varepsilon$ for all $n \geq n_2$.

Let for all m be a positive integer such that for all $m \geq n_1, n_2$.

Then $d(x, y) \leq d(x, x_m) + d(x_m, y)$

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$.

$\therefore\ d(x, y) < \varepsilon$.

Since $\varepsilon > 0$ is arbitary $d(x, y) = 0$.

$\therefore x = y$.

Theorem2.7:

let M be a metric space and $A \subseteq M$. Then

(vi) $x \in \bar{A}$iff there exists a sequence (x_n) in A such that $(x_n) \to x$.

(vii) $x$is a limit point of A iff there exists a sequence (x_n) of distinct points in A such that $(x_n) \to x$.

Proof:

Let $x \in \bar{A}$.

Then, $x \in A \cup D(A)$ (by theorem)

$\therefore x \in A$ or $x \in D(A)$

If $x \in A$, then the constant sequence $x, x, \ldots \ldots$ Is a sequence in A converging to x.

If $x \in D(A)$ then the open ball $B(x, 1/n)$ contains infinite number of points of A (by theorem)

$\therefore$ We can choose $x_n \in B(x, 1/n) \cap A$ such that $x_n \neq x_1, x_2, \ldots, x_{n-1}$ for each n.

$\therefore (x_n)$ is a sequence of distinct points in A. Also $d(x_n, x) < \frac{1}{n}$ for all n.

$\therefore \lim_{x \to \infty} d(x_n, x) = 0$.

$\therefore (x_n) \to x$.

Conversely, suppose there exists a sequence (x_n) in A such that $(x_n) \to x$

Then for any $r > 0$ there exists a positive integer n_0 such that $d(x_n, x) < r$ for all $n \geq n_0$.

$\therefore x_n \in B(x, r)$ for all $n \geq n_0$.

$\therefore B(x, r) \cap A \neq \phi$

$\therefore x \in \overline{A}$. (by corollary 2)

Further if (x_n) is a sequence of distinct points, $B(x, r) \cap A$ is infinite.

$\therefore x \in D(A)$.

$\therefore x$ is a limit point of A.

Definition: Let (M, d) be a metric space. let (x_n) be a sequence of points in M. (x_n) is said to be a caushy sequence in M if given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_0$.

Theorem 2.7:

Let (M, d) be a metric space. Then any convergent sequence in M is a Cauchy sequence.

Proof:

Let (x_n) be a convergent sequence of points in M converging to $x \in M$.

Let $\varepsilon > 0$ be given.

Then there exists a positive integer n_0 such that $(x_n, x) < \frac{1}{2}\varepsilon$ for all $n \geq n_0$.

Therefore, $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$for all $m, n \geq n_0$.

$= \varepsilon$for all $m, n \geq n_0$.

$\therefore$ $d(x_n, x_m) < \varepsilon$. for all $m, n \geq n_0$.

$\therefore$ $(x_n)$is a convergent sequence.

Note: The converse of the above theorem is not true.

Definition:A metric space M is said to be complete if every Caushy sequence in M converges to a point in M.

Theorem 2.8:

(Canton's Intersection Theorem)

Let M be a metric space. $M$is complete iff for every sequence (F_n) of nonempty closed subsets of M such that

$F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$and $d((F_n)) \to 0$. $\bigcap_{n=1}^{\infty} F_n$is nonempty.

Proof:

Let M be acomplete metric space.

Let (F_n) be a sequence of closed subsets of M such that

$F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ ------------ (1)

and $d((F_n)) \to 0$. ------------ (2)

we claim that . $\bigcap_{n=1}^{\infty} F_n$is nonempty.

For each positive integer n, choose a point $x_n \in F_n$.

By (1), $x_n, x_{n+1}, x_{n+2}, \ldots.$ all lies in F_n.

(i.e) $x_m \in F_n$for all $m \geq n$. ---------------------- (3)

Since $(d(F_n)) \to 0$, given $\varepsilon > 0$,there exists a positive integer n_0, such that $d(F_n) < \varepsilon$ for all $n \geq n_0$.

In particular $d(F_{n_0}) < \varepsilon$. ---------------- (4)

$\therefore d(x, y) < \varepsilon$for all $x, y \in F_n$.

Now,$x_m \in F_{n_0}$ for all $m \geq n_0$. (by(3))

$\therefore m, n \geq n_0 \Rightarrow x_m, x_n \in F_{n_0}$.

$\Rightarrow d(x_m, x_n) < \varepsilon$. (by(4))

$\therefore (x_n)$is a cauchy sequence in M.

Since M is complete there exists a point $x \in M$ such that $(x_n) \to x$.

We claim that $x \in \bigcap_{n=1} F_n$.

Now, for any positive integer n, $x_n, x_{n+1}, x_{n+2}, \ldots.$ is a sequence in F_n and this sequence converges to x.

$\therefore x \in \overline{F_n}$ (by theorem 3.2)

But $\overline{F_n}$ is closed and hence $\overline{F_n} = F_n$.

$\therefore x \in F_n$.

$\therefore x \in \bigcap_{n=1}^{\infty} F_n$. Hence $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

Conversely let,$(x_n)$is a cauchy sequence in$M$.

Let $F_1 = \{x_1, x_2, \ldots \ldots, x_n, \ldots.\}$

$F_1 = \{x_2, x_3, \ldots \ldots, x_n, \ldots.\}$

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$F_n = \{x_n, x_{n+1}, x_{n+2}, \ldots.\}$

Clearly $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$

$\therefore \overline{F_1} \supseteq \overline{F_2} \supseteq \cdots \supseteq \overline{F_n} \supseteq \cdots$

$\therefore (\overline{F_n})$is a decreasing sequence of closed of closed sets.

Now, since (x_n) is a Cauchy sequencegiven $\varepsilon > 0$ there exists a positive integer n_0, such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_0$.

∴ For any integer $n \geq n_0$, the distance between any two points of F_n is less than ε.

∴ $d(F_n) < \varepsilon$for all $n \geq n_0$

But $d(F_n) = d(\overline{F_n})$.

∴ $d(\overline{F_n}) < \varepsilon$for all $n \geq n_0$ ---------------------- (5)

$(d(\overline{F_n})) \to 0$.

Hence $\bigcap_{n=1}^{\infty} \overline{F_n}$ is nonempty

Let $x \in \bigcap_{n=1}^{\infty} \overline{F_n}$. Then x and $x_n \in \overline{F_n}$

∴ $d(x_n, x) \leq d(\overline{F_n})$.
∴ $d(x_n, x) < \varepsilon$for all $n \geq n_0$ (by(5))
∴ $(x_n) \to x$.
∴ $M$is complete.

Note:1 In the above theorem $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Note: 2 In the above theorem $\bigcap_{n=1}^{\infty} F_n$ may be empty if each F_n is not closed.

Note:3 In the above theorem $\bigcap_{n=1}^{\infty} F_n$ may be empty if the hypothesis $(d(\overline{F_n})) \to 0$is omitted.

Definition: A subset of a metric space M is said to be **nowhere dense** in Mif $Int\,\bar{A} = \phi$.

Definition:A subset of a metric space M is said to be of **first category** in M if A can be expressed as a countable union of nowhere dense sets.

A set which is not of first category is said to be of **second category**.

Theorem2.9:(Baire's Category Theorem)

Any complete metric space is of second category.

Proof: Let M be a complete metric space.

Claim:M is not of first category.

Since M is open and A_1 is nowhere dense,there exists an open ball say B_1 of radius less than 1 such that B_1 is disjoint from A_1. (refer theorem 3.6)

Let F_1 denote the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_1.

Now, $Int\ F_1$ is open and A_2 is nowhere dense.

∴ $Int\ F_1$contains an open ball B_2 of radius less than $\frac{1}{2}$ such that B_2 is disjoint from A_2.

Let F_2 be a concentric closed ball whose radius is $\frac{1}{2}$ times that of B_2. Now $Int\ F_2$ is open and A_3 is nowhere dense.

$\therefore$ $Int\ F_2$contains an open ball B_3 of radius less than $\frac{1}{4}$ such that B_2 is disjoint from A_3.

Let F_3 be a concentric closed ball whose radius is $\frac{1}{2}$ times that of B_3.

Proceeding like this we get a sequence of nonempty closed balls F_n such that $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$and $d(F_n) < \frac{1}{2^n}$.

Hence $(d(F_n)) \to 0$as $n \to \infty$.

Since M is complete, by Contor's intersection theorem,there exists a point x in M such that $x \in \bigcap_{n=1}^{\infty} F_n$.

Also each F_n is disjoint from A_n.

Hence, $x \notin F_n$ for all n .

$\therefore x \notin \bigcup_{n=1}^{\infty} A_n$.

$\therefore \bigcup_{n=1}^{\infty} A_n \neq M$. Hence M is of second category.

Corollary:R is of second category.

UNIT - III
COUNTINUITY

Definition:let(M_1, d_1) and (M_2, d_2) be metric spaces.

Let $f: M_1 \to M_2$ be a function. Let $a \in M_1$and $l \in M_2$. The function f is said to have a **limit** as $x \to a$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that,

$0 < d_1(x, a) < \delta \Rightarrow d_2(f(x), l) < \varepsilon.$

We write $\lim_{x \to a} f(x) = l.$

Definition: Let(M_1, d_1) and (M_2, d_2) be metric spaces.Let $a \in M_1$.A function $f: M_1 \to M_2$ is said to be **continuous** at a if given $\varepsilon > 0$, there exists $\delta > 0$ such that,

$d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon.$

$f$is said to be **continuous** if its continuous at every point of M_1.

Note:1f is continuous at a iff $\lim_{x \to a} f(x) = f(a)$.

Note:2 The condition $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$ can be rewritten as

(i) $x \in B(x, \delta) \Rightarrow f(x) \in B(f(a), \varepsilon)$ or

(ii) $f(B(a, \delta)) \subseteq B(f(a), \varepsilon)$.

Theorem 3.1:

Let (M_1, d_1) and (M_2, d_2) be metric spaces.Let $a \in M_1$. A function $f: M_1 \to M_2$ is continuous at a iff $(x_n) \to a \Rightarrow \big(f(x_n)\big) \longrightarrow f(a)$.

Proof: Suppose f is continuous at a.

Let (x_n) be a sequence in M_1 such that $(x_n) \to a$.

Claim:$\big(f(x_n)\big) \longrightarrow f(a)$.

Let $\varepsilon > 0$ be given. By definition of continuity, there exists $\delta > 0$ such that,

$d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$. -------------------- (1)

Since $(x_n) \to a$, there exists a positive integer n_0 such that $d_1(x_n, a) < \delta$ for all $n \geq n_0$.

$\therefore d_2(f(x), f(a)) < \varepsilon$for all $n \geq n_0$. (by(1))

$\therefore \big(f(x_n)\big) \longrightarrow f(a)$.

Conversely, suppose $(x_n) \to a \Rightarrow \big(f(x_n)\big) \longrightarrow f(a)$.

Claim:f is continuous at a.

Suppose f is not continuous at a. Then there exists an $\varepsilon > 0$ such that for all $\delta > 0$,

$f\big(B(a, \delta)\big) \not\subset B(f(a), \varepsilon)$

In particular, $f\left(B\left(a,\frac{1}{n}\right)\right) \not\subset B(f(a),\varepsilon)$.

Choose x_n such that $x_n \in B\left(a,\frac{1}{n}\right)$and $(x_n) \notin B(f(a),\varepsilon)$.

$\therefore d_1(x_n,a) < \frac{1}{n}$, and $d_2(f(x),f(a)) \geq \varepsilon$.

$(x_n) \to a$and$(f(x_n))$ not converges to $f(a)$ which is a contradiction to the hypothesis.

Hence, f is continuous at a.

Corollary 1:A function $f: M_1 \to M_2$ is continuous at a iff $(x_n) \to x \Rightarrow (f(x_n)) \longrightarrow f(x)$.

Theorem 3.2:

Let (M_1,d_1) and (M_2,d_2) be metric spaces. $f: M_1 \to M_2$is continuous iff $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

(i.e) f is continuous iff inverse image of every open set is open.

Proof:

Suppose f is continuous

Let G ba an opeb set in M_2.

Claim:$f^{-1}(G)$ is open in M_2.

If$f^{-1}(G)$ is empty, then it is open. Let $f^{-1}(G) \neq \phi$.

Let $x \in f^{-1}(G)$. Hence $f(x) \in G$.

Since G is open, there exists an open ball $B(f(x),\varepsilon)$ such that $B(f(x),\varepsilon) \subseteq G$.

Now, by definition of continuity, there exists an open ball $B(x,\delta)$ such that $f(B(x,\delta)) \subseteq B(f(x),\varepsilon)$.

$\therefore f(B(x,\delta)) \subseteq G$ (by(1))

$\therefore B(x,\delta) \subseteq f^{-1}(G)$

Since $x \in f^{-1}(G)$ is arbitrary, $f^{-1}(G)$ is open.

Conversely, suppose $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

we claim that f is continuous.

Let $x \in M_1$.

Now, $B(f(x),\varepsilon)$ is an open set in M_2.

$\therefore f^{-1}(B(f(x),\varepsilon)$is open in M_1 and $x \in f^{-1}(B(f(x),\varepsilon)$.

Therefore there exists $\delta > 0$ such that $B(x,\delta) \subseteq f^{-1}(B(f(x),\varepsilon)$.

$\therefore f(B(x,\delta)) \subseteq (B(f(x),\varepsilon)$.

$\therefore f$is continuous at x.

Since $x \in M_1$ is arbitrary f is continuous.

Theorem 3.3:

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f: M_1 \to M_2$ is continuous iff $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

Proof: Suppose $f: M_1 \to M_2$ is continuous.

Let $F \subseteq M_2$ be closed in M_2.

$\therefore F^c$is open in M_2.

$\therefore f^{-1}(F^c)$is open in M_1.

Conversely, suppose $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

We claim that f is continuous.

Let G be an open set in M_2.

$\therefore G^c$is open in M_2.

$\therefore f^{-1}(G^c)$is closed in M_1.

$\therefore [f^{-1}(G)]^c$is closed in M_1.

$\therefore f^{-1}(G)$is open in M_1.

$\therefore f$is continuous.

Theorem 3.4:

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f: M_1 \to M_2$ is continuous iff $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

Proof:

Suppose f is continuous.

Let $A \subseteq M_1$. Then $f(A) \subseteq M_2$.

Since f is continuous, $f^{-1}\left(\overline{f(A)}\right)$ is closed in M_1.

Also $f^{-1}\left(\overline{f(A)}\right) \supseteq A$ (since $\overline{f(A)} \supseteq f(A)$)

But $\bar{A}$ is the smallest closed set containing A.

$\therefore \bar{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$

$\therefore f(\bar{A}) \subseteq \overline{f(A)}$.

Conversely, let $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

To prove:f is continuous.

We shall show that if F is a closed set in M_2, then $f^{-1}(F)$ is closed in M_1.

By hypothesis, $f\left(\overline{f^{-1}(F)}\right) \subseteq \overline{ff^{-1}(F)}$

$\subseteq \bar{F}$.

$= F$. (since$F$ is closed.)

Thus $f\left(\overline{f^{-1}(F)}\right) \subseteq F$.

$\therefore \left(\overline{f^{-1}(F)}\right) \subseteq f^{-1}(F)$

Also $f^{-1}(F) \subseteq \left(\overline{f^{-1}(F)}\right)$.

$f^{-1}(F) = \overline{f^{-1}(F)}$.

Hence $f^{-1}(F)$ is closed.

$\therefore f$is continuous.

3.2Homeomorphism

Definition: Let Let(M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f: M_1 \to M_2$ is called a **homeomorphism** if

(i) $f$is 1-1 and onto.

(ii) $f$is continuous.

(iii) $f^{-1}$is continuous.

M_1andM_1 are said to be homeomorphic if there exists a homeomorphism$f: M_1 \to M_2$.

Definition: A function $f: M_1 \to M_2$ is said to be an open map if $f(G)$ is open in M_2 for every open set Gin M_1.

(ie) f is an open map if the image of an open set in M_1 is an openset in M_2.

$f$is called a closedmap if $f(F)$ is closed in M_2 for every closed set F in M_1.

Note: Let$f: M_1 \to M_2$ be a 1-1 onto function. Then f^{-1} is continuous iff f is an open map.

For, f^{-1} is continuous iff for any open set G in $M_1 (f^{-1})^{-1}(G)$ is open in M_2.

But, $(f^{-1})^{-1}(G) = f(G)$.

$\therefore f^{-1}$is continuous iff for every open set G in M_1, $f(G)$ is open in M_2.

$\therefore f^{-1}$is continuous iff f is an open map.

Note: Similarly f^{-1} is continuous iff f is a closed map.

Note: Let$f: M_1 \to M_2$ be a 1-1 onto map. Then the following are equivalent.

(i) $f$is homeomorphism.

(ii) $f$is continuous open map.

(iii) $f$is continuous closed map.

Proof:

(i)⇔(ii) follows from Note1 and the definition of homeomorphism.

(i))⇔(iii) follows from Note2 and the definition of homeomorphism.

Note: Let $f: M_1 \to M_2$ be a homeomorphism. $G \subseteq M_1$is open in M_1 iff $f(G)$ is open in M_2.

Note: Let$f: M_1 \to M_2$ be a 1-1 onto map. Then f is a homeomorphism iff it satisfies the following condition.

$F$is closed in M_1 iff $f(F)$ is closed in M_2.

3.3 Uniform Continuity

Definition : Let(M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f: M_1 \to M_2$ is said to be uniformly continuous on M_1 if given > 0 , there exists $\delta > 0$ such that,

$d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$.

Problem 3.5: Prove that $f: [0,1] \longrightarrow \boldsymbol{R}$ defined by $f(x) = x^2$ is uniformly continuous on $[0,1]$.

Solution:

Let $\varepsilon > 0$ be given. Let $x, y \in [0,1]$.

Then $|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y|$

$\leq 2|x - y|$ (since$x \leq 1$ and $y \leq 1$)

$\therefore |x - y| < \frac{1}{2}\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$.

$\therefore f$ is uniformly continuous on$[0,1]$.

Problem 3.6: Prove that the function $f: \boldsymbol{R} \longrightarrow \boldsymbol{R}$ defined by $f(x) = sin\, x$ is uniformly continuous on $\boldsymbol{R}$.

Solution:

Let $x, y \in \boldsymbol{R}$and $x > y$.

$sin\, x - sin y = (x - y) cos\, z$where$x > z > y$ (by mean value theorem)

$\therefore |sin\, x - sin\, y| = |x - y||cos\, z|$

$\leq |x - y|$ (since $|cos\, z| \leq 1$).

Hence for a given > 0 , we choose $\delta = \varepsilon$, we have $|x - y| < \delta \Longrightarrow |f(x) - f(y)| = |sin\, x - sin\, y| < \varepsilon$.

$\therefore f(x) = sin\, x$ is uniformly continuous on $\boldsymbol{R}$.

3.4 Discontinuous funtions on r

Definition: A function $f: \boldsymbol{R} \longrightarrow \boldsymbol{R}$ is said to approach to a **limit**l as x tends to a if given > 0 , there exists $\delta > 0$ such that

$0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon$and we write $\lim_{x \to a} f(x) = l$.

Definition: A function f is that to have l as the **right limit** at $x = a$ if given $\varepsilon > 0$, there exists

$\delta > 0$ such that $a < x < a + \delta \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim_{x \to a+} f(x) = l$.

Also we denote the right limit $l$by $f(a +)$.

A function f is that to have l as the **left limit** at $x = a$ if given > 0 , there exists $\delta > 0$ such that $a - \delta < x < a \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim_{x \to a-} f(x) = l$.

Also we denote the right limit $l$by $f(a -)$.

Note:$\lim_{x \to a} f(x) = l$iff $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = l$.

(i.e.) $\lim_{x \to a} f(x)$ exists iff the left and right limits of $f(x)$ at $x = a$ exists and are equal.

Note: The definition of continuity of f at $x = a$ can be formulated as follows.

$f$is continuous at at a iff $f(a +) = f(a -) = f(a)$.

Note: If $\lim_{x \to a} f(x)$ does not exists then one of the following happens.

(i) $\lim_{x \to a+} f(x)$does not exists.

(ii) $\lim_{x \to a-} f(x)$does not exists.

(iii) $\lim_{x \to a-} f(x)$and $\lim_{x \to a+} f(x)$ exist and are unequal.

Definition: If a function f is discontinuous at a then a is called a point of discontinuity for the function.

If a is a point of discontinuity of a function then any one of the following cases arises.

(i) $\lim_{x \to a} f(x)$exists but is not equal to $f(a)$.

(ii) $\lim_{x \to a-} f(x)$and $\lim_{x \to a+} f(x)$ exist and are not equal.

(iii) Either $\lim_{x \to a-} f(x)$ or $\lim_{x \to a+} f(x)$ does not exist.

Definition: let a be a point of disconitinuity for $f(x)$. $a$is said to be a point of discontinuity of the first kind if $\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ exist and both of them are finite and unequal.

$a$is said to be a point of discontinuity of the second kind if either $\lim_{x \to a-} f(x)$ or $\lim_{x \to a+} f(x)$ are does not exist.

Definition: Let $A \subseteq R$. Afunction $f: A \to \boldsymbol{R}$ is called monotonic increasing if $x, y \in A$and $x < y \Rightarrow f(x) \leq f(y)$.

f is called monotonic decreasing if $x, y \in A$ and $x > y \Rightarrow f(x) \geq f(y)$.

$f$is called monotonic if it is either monotonic increasing or monotonic decreasing.

Theorem 3.7:

Let $f:[a,b] \to \boldsymbol{R}$be a **monotionic increasing function.** Then has a left limit and right limit at every point (a,b). Also f has a right limit at a and$f$ has a left limit at b. Further x<y ⇒ $f(x+) \leq f(y-)$.

Similar result is true for monotonic decreasing function.

Proof:

Let $f:[a,b] \to \boldsymbol{R}$ **be a monotionic increasing function.**

Let x∈[a,b]. then $\{f(t)/a \leq t < x\}$ is bounded above by $f(x)$.

Let $l = l.u.b\{f(t)/a \leq t < x\}$

Claim: $f(x-) = l$

Let $\varepsilon > 0$ be given .By definition $l.u.b$ there exists t such that $a \leq t < x$ and l- $-\varepsilon < f(t) \leq l$

Therefore $t < u < x \Rightarrow l - \varepsilon < f(t) \leq f(u) \leq l$

(since f is monotonic increasing)

$\Rightarrow l - \varepsilon < f(u) \leq l$

$\therefore\ \boldsymbol{x - \delta < u < x \Rightarrow l - \varepsilon < f(u) \leq l}$**where**$\boldsymbol{\delta = x - t}$

$\therefore \mathbf{f(x-) = l}$

Similarly we can prove that $\boldsymbol{f(x+) = g.l.b\{f(t)/x < t \leq b\}}$

To Prove :$\boldsymbol{x < y \Rightarrow f(x+) \leq f(y-)}$

Let $\boldsymbol{x < y}$

Now ,$\boldsymbol{f(x+)}$ = $\boldsymbol{g.l.b\{f(t)/x < t \leq b\}}$

= $\boldsymbol{g.l.b\{f(t)/x < t \leq y\}}$

(since$\boldsymbol{f}$ is monotonic increasing)

Also, $\boldsymbol{f(y-)}$ = $\boldsymbol{l.u.b\{f(t)/a \leq t < y\}}$

= $\boldsymbol{l.u.b\{f(t)/x \leq t < y\}}$

$\boldsymbol{f(x+) \leq, f(y-)}$

The proof of monotonic decreasing function is similar.

Theorem 3.8:

Let$\boldsymbol{f:[a,b] \to R}$ **be a monotionic function. Then the set of points of [a,b] at which $\boldsymbol{f}$ is discontinuous is countable.**

Proof:

Let E = $\{\boldsymbol{x/x \in [a,b]\ and\ f}$ is discontinuous at x}

Let $\boldsymbol{x} \in \boldsymbol{E}$. then by previous theorem,

$\boldsymbol{f}(\boldsymbol{x}+)$ ***and*** $\boldsymbol{f}(\boldsymbol{x}-)$ ***exists and*** $\boldsymbol{f}(\boldsymbol{x}-) \leq \boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\boldsymbol{x}+)$

If $\boldsymbol{f}(\boldsymbol{x}-) = \boldsymbol{f}(\boldsymbol{x}+)$ then $\boldsymbol{f}(\boldsymbol{x}-) = \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}+)$

$\therefore$ $\boldsymbol{f}$is continuous at $\boldsymbol{x}$ which is a contradiction.

$\therefore$ $\boldsymbol{f}(\boldsymbol{x}-) \neq \boldsymbol{f}(\boldsymbol{x}+)$

$\therefore \boldsymbol{f}(\boldsymbol{x}-) < \boldsymbol{f}(\boldsymbol{x}+)$

Now choose a rational number $\boldsymbol{r}(\boldsymbol{x})$ such that $\boldsymbol{f}(\boldsymbol{x}-) < \boldsymbol{r}(\boldsymbol{x}) < \boldsymbol{f}(\boldsymbol{x}+)$.

This define a map $\boldsymbol{r}$ from $\boldsymbol{E}$ to $\boldsymbol{Q}$ which maps $\boldsymbol{x}$ to r$(\boldsymbol{x})$.

Claim: $\boldsymbol{r}$ is 1-1

Let $\boldsymbol{x_1} < \boldsymbol{x_2}$

$\therefore \boldsymbol{f}(\boldsymbol{x_1}+) < \boldsymbol{f}(\boldsymbol{x_2}-)$ (by previous theorem)

Also, $\boldsymbol{f}(\boldsymbol{x_1}-) < \boldsymbol{r}(\boldsymbol{x_1}) = \boldsymbol{f}(\boldsymbol{x_1}+)$

And $\boldsymbol{f}(\boldsymbol{x_2}-) < \boldsymbol{r}(\boldsymbol{x_2}) = \boldsymbol{f}(\boldsymbol{x_2}+)$.

$\therefore$ $\boldsymbol{r}(\boldsymbol{x_1}) < \boldsymbol{f}(\boldsymbol{x_2}+) < \boldsymbol{f}(\boldsymbol{x_2}-) < \boldsymbol{r}(\boldsymbol{x_2})$.

Thus $\boldsymbol{x_1} < \boldsymbol{x_2} \Rightarrow$ $\boldsymbol{r}(\boldsymbol{x_1}) < \boldsymbol{r}(\boldsymbol{x_2})$.

Therefore,$\boldsymbol{r}$: $\boldsymbol{E} \rightarrow \boldsymbol{Q}$ is 1-1.Hence $\boldsymbol{E}$ is countable

UNIT - IV
CONNECTEDNESS

Definition: Let (M,d) be a metric space. $M$is said to be **connected** if M cannot be represented as the union of two disjoint nonempty open sets.
If M is not connected it is to be **disconnected**.

Example: Let $M = [1,2] \cup [3,4]$ with usual metric. Then M is disconnected.
Proof:
$[1,2]$and$[3,4]$ are open in M.
Thus, M is the union of two disjoint nonempty open dets namely $[1,2]$and $[3,4]$.
Hence M is disconnected.

Theorem 4.1:
Let (M,d) be a metric space. Then the following are equivalent.
$i)$ M is connected.
$ii)$ M cannot be written as the union of two disjoint nonempty closed sets.
$iii)$ M cannot be written as the union of two nonempty sets A and B such that $A \cap \bar{B} = \bar{A} \cap B = \phi$.
$iv)$ M and ϕ are the only sets which are both open and closed in M.
Proof:
(i)⇒(ii)
Suppose (ii) is true.
$\therefore M = A \cup B$where$A$ and B are closed $A \neq \phi, B \neq \phi$ and $A \cap B = \phi$.
$\therefore A^c = B$and $B^c = A$.
Since A and B are closed, A^c and B^c are open.
$\therefore B$and$A$ are open.
Thus M is the union of two disjoint nonempty open sets.
$\therefore M$is not connected which is a contradiction.
∴ (i)⇒(ii)
(ii)⇒(iii)
Suppose (iii) is not true.
Then $M = A \cup B$ where $A \neq \phi, B \neq \phi$and $A \cap \bar{B} = \bar{A} \cap B = \phi$.
Claim: A and B are closed.
Let $x \in \bar{A}$.
$\therefore x \notin B$ (since $\bar{A} \cap B = \phi$)
$\therefore x \in A$ (since $A \cup B = M$)
$\bar{A} \subseteq A$.
But $A \subseteq \bar{A}$.
$\therefore A = \bar{A}$and hence A is closed.
Similarly B is closed.

Now, $A \cap B = \bar{A} \cap B$. (since $A = \bar{A}$).
$= \phi$.
Thus $M = A \cup B$where $A \neq \phi, B \neq \phi, A\ and\ B$ are closed and $A \cap B = \phi$ which is contradiction to (ii).
∴(ii)⇒(iii)
(iii) ⇒(iv)
Suppose (iv) is not true.
Then there exists $A \subseteq M$ such that $A \neq M$ such that $A \neq M$ and $A \neq \phi$ and A is both open and closed.
Let $B = A^c$.
Then B is also both open and closed and $B \neq \phi$.
Also $M = A \cup B$.
Further $\bar{A} \cap B = A \cap A^c$. (since$A = \bar{A}$ and $A = A^c$)
$= \phi$.
Similarly $A \cap \bar{B} = \phi$.
∴ $M = A \cup B$where$A \cap \bar{B} = \phi = \bar{A} \cap B$ which is a contradiction to (iii).
∴(iii)⇒(iv).
(iv)⇒(i).
Suppose M is not connected.
∴ $M = A \cup B$ where$A \neq \phi, B \neq \phi, A\ and\ B$ are open and $A \cap B = \phi$.
Then $B^c = A$.
Now, since B is open A is closed.
Also $A \neq \phi$and $A \neq M$. (since $B \neq \phi$)
∴ $A$is a proper non empty subset of M which is both open and closed which is a contradiction to (iv).
∴ (iv))⇒(i).

Theorem 4.2
A metric space M is connected iff there does not exist a continuous function f from M onto the discrete metric space $\{0,1\}$.
Proof: Suppose there exists a continuous function f from Monto $\{0,1\}$.
Since $\{0,1\}$ is discrete,$\{0\}$ and $\{1\}$ are open.
∴ $A = f^{-1}(\{0\})$and$B = f^{-1}(\{1\})$ are open in M.
Since f is onto, A and B are non empty.
Clearly $A \cap B = \phi$and $A \cup B = M$.
Thus $M = A \cup B$ where A and B are disjoint nonempty open sets.
∴ $M$is not connected which is a contradiction.
Hence there does not exist a continuous function from onto the discrete metric space $\{0,1\}$.
Conversely, suppose M is not connected.
Then,there exists a disjoint nonempty open sets A and B in M such that $M = A \cup B$.

Now, define $f: M \to \{0,1\}$ by $f(x) = \begin{cases} 0 \; if \; x \in A \\ 1 \; if \; x \in B \end{cases}$

Clearly f is onto.

Also, $f^{-1}(\phi) = \phi, f^{-1}(\{0\}) = a, f^{-1}(\{1\}) = B \; and \; f^{-1}(\{0,1\}) = M.$

Thus the inverse image of every open set in $\{0,1\}$ is open in M.

Hence f is continuous.

Thus there exists a continuous function f from M onto $\{0,1\}$.which is a contradiction.

Hence M is not connected.

Problem 4.3:

Let M be a metric space. Let A be a connected subset of M. If B is a subset of of M such that $A \subseteq B \subseteq \bar{A}$ then B is connected. In particular $\bar{A}$ is connected.

Solution: Suppose B is not connected.

Then $B = B_1 \cup B_2$ where $B_1 \neq \phi, B_2 \neq \phi, B_1 \cap B_2 = \phi$ and B_1 and B_2 are open in B.

Now, since B_1 and B_2 are open sets in B there exists open sets G_1 and G_2 in M such that $B_1 = G_1 \cap B \; and \; B_2 = G_2 \cap B$.

$\therefore B = B_1 \cup B_2 = (G_1 \cap B) \cup (G_2 \cap B) = (G_1 \cup G_2) \cap B.$

$\therefore B \subseteq G_1 \cup G_2.$

$\therefore A \subseteq G_1 \cup G_2$ (since $A \subseteq B$)

$\therefore A = (G_1 \cup G_2) \cap A.$

$= (G_1 \cap A) \cup = (G_1 \cap A).$

Now, $G_1 \cap A$ and $G_2 \cap A$ are open in A.

Further, $(G_1 \cap A) \cup (G_2 \cap A) = (G_1 \cup G_2) \cap A.$

$= (G_1 \cup G_2) \cap B$ (since $A \subseteq B$)

$= (G_1 \cap B) \cap (G_2 \cap B)$

$= B_1 \cap B_2.$

$= \phi.$

$\therefore (G_1 \cap A) \cup (G_2 \cap A) = \phi.$

Now, since A is connected, either $G_1 \cap A = \phi \; or \; G_2 \cap A = \phi.$

Without loss of generality let us assume that $G_1 \cap A = \phi.$

Since G_1 is open in M, we have $G_1 \cap \bar{A} = \phi.$

$\therefore G_1 \cap B = \phi.$ (since $B \subseteq \bar{A}$)

$\therefore B_1 = \phi$which is a contradiction.

Hence B is not connected.

4.2 Connected Subsets of $\boldsymbol{R}$

Theorem 4.4:

A subspace of $\boldsymbol{R}$ is connected iff it is an interval.

Proof:

Let A be a connected subset of $\boldsymbol{R}$.

Suppose$A$ is not an interval.

Then there exists $a, b, c \in \boldsymbol{R}$such that, $a < b < c$ and $a, c \in A$but $b \notin A$.
Let $A_1 = (-\infty, b) \cap A$and $A_2 = (b, \infty) \cap A$.
Since $(-\infty, b)$ and (b, ∞) are open in $\boldsymbol{R}$, A_1 and A_2 are open sets in A.
Also, $A_1 \cap A_2 = \phi$and $A_1 \cup A_2 = A$.
Further $a \in A_1$and $c \in A_2$.
Hence $A_1 \neq \phi$and $A_2 \neq \phi$.
Thus A is the union of two disjoint nonempty open sets $A_1$and A_2.
Hence A is not connected which is a contradiction.
Hence A is an interval.
Conversely, let A be an interval.
Claim:A is connected.
Suppose A is not connected.
Let $A = A_1 \cup A_2$ where $A_1 \neq \phi, A_2 \neq \phi, A_1 \cap A_2 = \phi$ and A_1 and A_2 are closed in A.
Choose $x \in A_1$and $z \in A_2$.
Since $A_1 \cap A_2 = \phi$ we have $x \neq z$.
Without loss of generality let us assume that $x < z$.
Now, since A is an interval we have $[x, z] \subseteq A$.
(i.e) $[x, z] \subseteq A_1 \cup A_2$.
∴ Every element of $[x, z]$ is either in A_1 or in A_2.
Now, let $y = l.u.b.\{[x, z] \cap A_1\}$.
Clearly $x \leq y \leq z$.
Hence $y \in A$.
Let $\varepsilon > 0$ be given. Then by the definition of $l.u.b.$ there exists $t \in [x, z] \cap A_1$ such that $y - \varepsilon < t \leq y$.
$\therefore (y - \varepsilon, y + \varepsilon) \cap ([x, z] \cap A_1) \neq \phi$.
$\therefore y \in \overline{[x, z] \cap A_1}$
$\therefore y \in [x, z] \cap A_1$
$\therefore y \in A_1$.
Again by the definition of y, $y + \varepsilon \in A_2$ for all $\varepsilon > 0$ such that $y + \varepsilon \leq z$.
$\therefore y \in \overline{A_2}$
$\therefore y \in A_2$ (sinceA_2 is closed)
$\therefore y \in A_1 \cap A_2$ [by(1) and (2)] which is a contradiction since $A_1 \cap A_2 = \phi$.
Hence A is connected.

Theorem 4.5:
$\boldsymbol{R}$ is connected.
Proof:$\boldsymbol{R} = (-\infty, \infty)$ is an interval.
∴ $\boldsymbol{R}$is connected.

4.3 Connectedeness and Continuity
Theorem 4.6:

let M_1 be a connected metric space. Let M_2 be any metric space. Let $f: M_1 \to M_2$ be a continuous function. Then $f(M_1)$ is a connected subset of M_2.
(i.e) Any continuous image of a connected set is connected.

Proof:

Let $f(M_1) = A$ so that f is function on M_1onto A.

Claim:A is connected.

Suppose A is not connected. Then there exists a proper non empty subset of B of A which is both open and closed in A.

$\therefore f^{-1}(B)$is a proper nonempty subset of M_1 which is both open and closed in M_1.

Hence M_1 is not connected which is contradiction.

Hence A is connected.

Theorem 4.7:

let f be a real valued continuous function defined on an interval I. Then f takes every value between any two values it assumes. (This is known as **the intermediate value theorem**)

Proof:

Let $a, b \in I$and $f(a) \neq f(b)$.

Without loss of generality we assume that $f(a) < f(b)$.

Let c be such that $f(a) < c < f(b)$.

The interval I is a connected subset of $\boldsymbol{R}$.

$\therefore f(I)$is a connected subset of $\boldsymbol{R}$. (by theorem 4.6)

$\therefore f(I)$is an interval. (by theorem 4.6)

Also $f(a), f(b) \in f(I)$. Hence $[f(a), f(b)] \subseteq f(I)$.

$\therefore c \in f(I)$ (since $f(a) < c < f(b)$)

$\therefore c = f(x)$for some $x \in I$.

4.2 Compact Metric Spaces

Definition: Let M be a metric space. A family of opensets $\{G_\alpha\}$ in M is called an open cover for Mif $\cup G_\alpha = M$.

A subfamily of $\{G_\alpha\}$ whichitself is an open cover is called a **subcover**.

A metric space M is said to be **compact** if every open cover for M has finite subcover.

(i.e) for each family of open sets $\{G_\alpha\}$ such that $\cup G_\alpha = M$, there exists a finite subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \dots \dots, G_{\alpha_n}\}$ such that $\cup_{i=1}^{n} G_{\alpha_i} = M$.

Theorem 4.8:

Let M be a metric space. Let $A \subseteq M$. $A$is compact iff given a family of open sets $\{G_\alpha\}$ in M such that $\cup G_\alpha \supseteq A$ there exists a subfamily
$G_{\alpha_1}, G_{\alpha_2}, \dots \dots, G_{\alpha_n}$such that $\cup_{i=1}^{n} G_{\alpha_i} \subseteq A$.

Proof:

Let A be a compact subset of M.

Let $\{G_\alpha\}$ be a family of open sets in M such that $\cup G_\alpha \supseteq A$.

Then $(\cup G_\alpha) \cap A = A$.
$\therefore \cup (G_\alpha \cap A) = A$.
Also $G_\alpha \cap A$ is open in A.
$\therefore$ The family $\{G_\alpha \cap A\}$ is an open cover for A.
Since A is compact this open cover has a finite subcover, say, $G_{\alpha_1} \cap A, G_{\alpha_2} \cap A, \dots \dots ., G_{\alpha_n} \cap A$.
$\therefore \cup_{i=1}^{n}(G_{\alpha_i} \cap A) = A$.
$\therefore (\cup_{i=1}^{n} G_{\alpha_i}) \cap A = A$.
$\therefore \cup_{i=1}^{n} G_{\alpha_i} \subseteq A$.
Conversely let $\{H_\alpha\}$ be an open cover for A.
$\therefore$ Each H_α is open in A.
$\therefore H_\alpha = G_\alpha \cap A$ where G_α is open in M.
Now, $\cup H_\alpha = A$.
$\therefore \cup (G_\alpha \cap A) = A$.
$\therefore (\cup G_\alpha) \cap A = A$.
$\therefore \cup G_\alpha \supseteq A$.
Hence by hypothesis there exists a finite subfamily $G_{\alpha_1}, G_{\alpha_2}, \dots \dots ., G_{\alpha_n}$ such that $\cup_{i=1}^{n} G_{\alpha_i} \subseteq A$.
$\therefore (\cup_{i=1}^{n} G_{\alpha_i}) \cap A = A$.
$\therefore \cup_{i=1}^{n}(G_{\alpha_i} \cap A) = A$.
$\therefore \cup_{i=1}^{n} H_{\alpha_i} = A$.
Thus $\{H_{\alpha_1}, H_{\alpha_2}, \dots \dots ., H_{\alpha_n}\}$ is a finite subcover of the open cover $\{H_\alpha\}$.
$\therefore A$ is compact.

Theorem 4.9:
Any compact subset A of a metric space M is bounded.
Proof:
Let $x_0 \in A$.
Consider $\{B(x_0, n) | n \in N\}$.
Clearly $\cup_{i=1}^{n} B(x_0, n) = M$.
$\therefore \cup_{i=1}^{n} B(x_0, n) \supseteq A$.
Since A is compact there exists a finite subfamily say, $B(x_0, n_1), B(x_0, n_2), \dots \dots \dots, B(x_0, n_k)$ such that $\cup_{i=1}^{k} B(x_0, n_1) \supseteq A$.
Let $n_0 = \max\{n_1, n_2, \dots \dots ., n_k\}$.
Then $\cup_{i=1}^{k} B(x_0, n_i) = B(x_0, n_0)$.
$\therefore B(x_0, n_0) \supseteq A$.
We know that $B(x_0, n_0)$ is a bounded set and a subset of a bounded set is bounded.
Hence A is bounded.
Theorem 4.10:
Any compact subset A of a metric space (M, d) is closed.
Proof:
To prove: A is closed.

We shall prove that A^c is open.
Let $y \in A^c$ and let $x \in A$. Then $x \neq y$.
$\therefore d(x,y) = r_x > 0$.
It can be easily verified that $B\left(x, \frac{1}{2}r_x\right) \cap B\left(y, \frac{1}{2}r_x\right) = \phi$.
Now consider the collection $\{B\left(x, \frac{1}{2}r_x\right) / x \in A\}$.
Clearly $\bigcup_{x\in A} B\left(x, \frac{1}{2}r_x\right) \supseteq A$.
Since A is compact there exists a finite number of such open balls say,
$B\left(x_1, \frac{1}{2}r_{x_1}\right), \ldots\ldots, B(x_n, \frac{1}{2}r_{x_n})$ such that $\bigcup_{i=1}^{n} B(x_i, \frac{1}{2}r_{x_i}) \supseteq A$. ----------- (1)
Now, let $V_y = \bigcap_{i=1}^{n} B\left(y, \frac{1}{2}r_x\right)$.
Clearly V_y is an open set containing y.
Since $B\left(y, \frac{1}{2}r_y\right) \cap \left(x, \frac{1}{2}r_x\right) = \phi$, we have $V_y \cap B(x, \frac{1}{2}r_{x_i}) = \phi$ for each $i = 1,2, \ldots \ldots, n$.
$\therefore V_y \cap [\bigcup_{i=1}^{n} B(x, \frac{1}{2}r_{x_i})] = \phi$.
$\therefore V_y \cap A = \phi$. (by (1)).
$\therefore V_y \subseteq A^c$.
$\therefore \bigcup_{y\in A^c} V_y = A^c$ and each V_y is open.
$\therefore A^c$ is open. Hence A is closed.

Theorem 4.11:
A closed subspace of a compact metric space is compact.
Proof:
Let M be a compact metric space.
Let A be a nonempty closed subset of M.
Claim: A is compact.
Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in M such that, $\bigcup_{\alpha\in I} G_\alpha \supseteq A$.
$\therefore A^c \cup [\bigcup_{\alpha\in I} G_\alpha] = M$.
Also A^c is open. (since A is closed).
$\therefore \{G_\alpha/\alpha \in I\} \cup \{A^c\}$ is an open cover for M.
Since M is compact it has a finite subcover say, $G_{\alpha_1}, G_{\alpha_2}, \ldots \ldots, G_{\alpha_n}, A^c$.
$\therefore (\bigcup_{i=1}^{n} G_{\alpha_i}) \cup A^c = M$.
$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.
$\therefore A$ is compact.

4.3 Compact Subsets of $\boldsymbol{R}$.
Theorem 4.12: (Heine Borel Theorem)
Any closed interval $[a,b]$ is a compact subset of $\boldsymbol{R}$.
Proof:
Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in $\boldsymbol{R}$ such that $\bigcup_{\alpha\in I} G_\alpha \supseteq [a,b]$.
Let $S = \{x | x \in [a,b]\ and\ [a,x] can\ be\ covred\ by\ \ a\ finite\ number\ of\ G_\alpha' s\}$.

Clearly $a \in S$ and hence $S \neq \phi$.
Also S is bounded above by b.
Let c denote the $l.u.b.$of S.
Clearly $c \in [a,b]$.
$\therefore c \in G_{\alpha_1}$for some $\alpha_1 \in I$.
Since G_{α_1} is open, there exists $\varepsilon > 0$ such that $(c-\varepsilon, c+\varepsilon) \subseteq G_{\alpha_1}$.
Choose $x_1 \in [a,b]$ such that $x_1 < c$and $[x_1, c] \subseteq G_{\alpha_1}$.
Now, since $x_1 < c$, $[a, x_1]$ can be covered by a finite number of $G_\alpha{}'s$.
These finite number of $G_\alpha{}'s$ together with G_{α_1}covers $[a,c]$.
$\therefore$ By definition of $S, c \in S$.
Now, we claim that $c = b$.
Suppose $c \neq b$.
Then choose $x_2 \in [a,b]$ such that $x_2 > c$and $[c, x_2] \subseteq G_{\alpha_1}$.
As before,$[a, x_2]$ can be covered by a finite number of $G_\alpha{}'s$.
Hence $x_2 \in S$.
But $x_2 > c$ which is a contradiction, since c is the $l.u.b.$of S.
$\therefore c = b$.
$\therefore$ $[a,b]$can be covered by a finite number of $G_\alpha{}'s$.
$\therefore$ $[a,b]$is a compact subset of $\boldsymbol{R}$.

Theorem 4.13:
Asubset of $\boldsymbol{R}$ is compact iff A is closed and bounded.
Proof:
If A is compact then A is closed and bounded.
Conversely, let A be a subset of $\boldsymbol{R}$ which is closed and bounded.
Since A is bounded we can find a closed interval $[a,b]$ such that $A \subseteq [a,b]$.
Since A is closed in $\boldsymbol{R}$, A is closed in $[a,b]$ also.
Thus A is a closed subset of the compact space $[a,b]$.
Hence A is compact. (by theorem 4.11)

UNIT - V
RIEMAN INTEGRAL

If I is the integral of real number,the length of I is denoted by $|I|$.

Set of measure Zero:

A subset $E \subset R$ is said to be a measure Zero if for each $\varepsilon > 0$, there exists a finite (or) countable number of open intervals, $I_1, I_2, \dots \dots ..$ such that $E \subset \bigcup_{n=1}^{\infty} I_n$.
$\sum_{n=1}^{\infty} |I_n| < \varepsilon$.

Derivatives:

Let f be a real valued function defined on an Interval $[a, b] \subseteq R$. It is derivable at an interior point $c \in (a, b)$.

(i) If $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists.
$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$exists.
Where $x = c + h \to x - c = h$.

(ii) $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$is called the left hand derivative $= Lf'(c)$.

(iii) $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ is called the right hand derivative $= Rf'(c)$

(iv) If $f'(c) = Lf'(c) = Rf'(c)$ then we say $f(x)$is derivable.

(v) $f'(a) = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$.

(vi) $f'(b) = \lim_{x \to b^-} \frac{f(x)-f(b)}{x-b}$.

Example 1:

Show that the function $f(x) = x^2$ is derivable in $[0,1]$.

Solution:

(i) Let $x_0 \in (0,1)$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}.$$
$$= \lim_{x \to x_0} \frac{x^2-{x_0}^2}{x-x_0}.$$
$$= \lim_{x \to x_0} \frac{(x+x_0)(x-x_0)}{x-x_0}.$$
$$= \lim_{x \to x_0} (x + x_0) = x_0 + x_0 = 2x_0.$$

∴derivable exists an interior point.

(ii) $$f'(0) = \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0}.$$
$$= \lim_{x \to 0^+} \frac{x^2-0}{x-0}.$$
$$= \lim_{x \to 0^+} \frac{x^2}{x}.$$
$$= \lim_{x \to 0^+} x = 0.$$

∴ $f'(0)$exists.

(iii) $f'(1) = \lim_{x\to f} \frac{f(x)-f(1)}{x-1}$.

$= \lim_{x\to f} \frac{x^2-1}{x-1}$.

$= \lim_{x\to f} \frac{(x+1)(x-1)}{(x-1)}$.

$= \lim_{x\to f}(x+1) = 1 + 1 = 2$.

$\therefore f'(1)$exists.

Hence $f(x)$ is differentiable in the closed interval $(0,1)$.

Example 2:

A function f is defined on R where $f(x) = \begin{cases} x \ if \ 0 \leq x < 1 \\ 1 \ if \ x \geq 1 \end{cases}$. Discuss the derivability at $x = 1$.

Solution:

$Lf'(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$.

$= \lim_{x\to 1^-} \frac{x-1}{x-1}$.

$= \lim_{x\to 1^-} 1$.

$\therefore Lf'(1) = 1$.

$Rf'(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$.

$= \lim_{x\to 1^+} \frac{1-1}{x-1}$.

$= 0$.

$\therefore Rf'(1) = 0$.

$Lf'(1) \neq Rf'(1)$.

(i.e.) $f'(1)$ does not exists.

$f$is not derivable at $x = 1$.

Example 3:

Discuss the derivability of $f(x) \ at \ 0, f(x) = |x|$.

Solution:

$Lf'(0) = \lim_{x\to 0^-} \frac{f(x)-f(1)}{x-0}$.

$= \lim_{x\to 0^-} \frac{-x-0}{x}$.

$= \lim_{x\to 0^-} \frac{-x}{x}$

$= \lim_{x\to 0^-} 1$.

$Lf'(0) = -1$.

$Rf'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$.

$= \lim_{x\to 0^+} \frac{x-0}{x-0}$.

$= \lim_{x\to 0^+} \frac{1}{1} = 1$.

$\therefore Rf'(1) = 1.$

$Lf'(1) \neq Rf'(1).$

(i.e.) $f'(0)$ does not exists.

$f$is not derivable at $x = 0$.

Example 4:

$f(x) = \begin{cases} x^2 \, sinx \frac{1}{x} \; if \; x \neq 0 \\ \quad 0 \; if \; x = 0 \end{cases}.$

Prove that f is derivable at $x = 0$but $\lim_{x \to 0} f'(x) \neq f'(0)$.

Solution:

$$Lf'(0) = \lim_{x \to 0^-} \frac{f(x)-f(1)}{x-0}$$
$$= \lim_{x \to 0^-} \frac{x^2 \, sin\frac{1}{x} - 0}{x}.$$
$$= \lim_{x \to 0^-} \frac{x^2}{x} sin \, \frac{1}{x}.$$
$$= \lim_{x \to 0^-} \; sin \frac{1}{0} = 0.$$

$Lf'(0) = 0.$

$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0}.$$
$$= \lim_{x \to 0^+} \frac{x^2 \, sin\frac{1}{x} - 0}{x-0}.$$
$$= \lim_{x \to 0^+} x^2 \, sin \frac{1}{x}.$$
$$= \lim_{x \to 0^+} \; sin \frac{1}{0}$$

$\therefore Rf'(1) = 0.$

$Lf'(1) = Rf'(1).$

Hence f is not derivable at $x = 0$.

Theorem:

A function which is derivable at a point is necessarily continuous at that point.

Proof:

Let a function f be derivable at $x = c$.

Then $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exist.

To prove: f is continuous at$x = c$.$f(x) - f(c) = \frac{f(x)-f(c)}{x-c} \times (x - c)$

$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} [\frac{f(x)-f(c)}{x-c} (x - c)].$

$= [\lim_{x \to c} \frac{f(x)-f(c)}{x-c}][\lim_{x \to c} (x - c)].$

$\lim_{x \to c} [f(x) - f(c)] = 0.$

$\lim_{x \to c} f(x) - \lim_{x \to c} f(c) = 0.$

$\lim_{x \to c} f(x) = \lim_{x \to c} f(c).$

$\therefore \lim_{x \to c} f(x) = f(c)$.
$\therefore f$ is continuous in $x = c$.

Note:

Converse of this theorem need not be true.

Rolle's theorem:

If a function f defined on $[a, b]$ is,

(i) Continuous on $[a, b]$.
(ii) Derivable on (a, b).
(iii) $f(a) = f(b)$ then there exists one real number c between $a \times b[a < c < b]$ such that $f'(c) = 0$.

Proof:

Since the function is continuous on $[a, b]$, it is bounded.
Let m and M are the infremum (g.l.b) and supremum (l.u.b) respectively of the function f then there exists points c and d in $[a, b]$ such that $f(c) = m$ and $f(d) = M$.
Case (i):
Let $m = M$, then f is constant.
$f(x) = M \; for \; all \; x \in [a, b]$.
$\therefore f(x) = 0 \; for \; all \; x \in [a, b]$.
For $c \in (a, b), f(c) = m$, that is $f'(c) = 0 \; for \; all \; c \in (a, b)$.
Case (ii):
Let $m \neq M$.
Now both m and M cannot be equal to $f(a)$.
$f(c) = m \neq f(a) \Rightarrow c \neq a$.
Similarly, $f(c) = M \neq f(b) \Rightarrow c \neq b$.
$\Rightarrow c \in (a, b)$.
Claim: $f'(c) = 0$.
If $f'(c) < 0$,there exists $(c, c + \delta_1)$ such that $f(x) < f(c) = M \; for \; all \; x, x \in (c, c + \delta_1)$.
Which is a contradiction.
If $f'(c) > 0$,there exists $(c - \delta_1, c)$ such that $f(x) < f(c) = M \; for \; all \; x, x \in (c - \delta_1, c)$.
Which is a contradiction.
Hence,$f'(c) = 0$.

Legrange's Mean Value Theorem

If a function f defined on $[a, b]$ is,

(i) Continuous on $[a, b]$.
(ii) Derivable on (a, b).

$f(a) = f(b)$then there exists one real number c between $a \times b[a < c < b]$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof:

Let $\phi(x) = f(x) + Ax$ where A is a constant such that $\phi(a) = \phi(b)$.
Then $f(a) + Aa = f(b) + Ab$.
$A(b-a) = f(a) - f(b)$.
$= -[f(b) - f(a)]$
$A = \frac{-[f(b)-f(a)]}{b-a}$.
Since $\phi(x)$ is a sum of two continuous and derivable function.
(i) $\phi$is continuous on $[a, b]$.
(ii) $\phi$is derivable on $[a, b]$.
(iii) $\phi(a) = \phi(b)$.
Therefore by Rolle's theorem, there exists $c \in (a, b)$ such that $\phi'(c) = 0$.
(i.e) $f'(c) + A = 0$.
$f'(c) = -A$.
$f'(c) = \frac{f(b)-f(a)}{b-a}$.

Cauchy's Mean Value Theorem:
If two functions f, g defined on $[a, b]$ are
(i) Continuous on $[a, b]$.
(ii) Derivable on $[a, b]$.
(iii) $g'(x) \neq 0$ for any $x \in (a, b)$ then there exists one real number c between $a\ and\ b$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

The Fundamental Theorem of Calculus:
A function f is bounded and integrable on $[a, b]$ and there exists a function f such that $f' = f\ on\ [a, b]$. Then $\int_a^b f\ dx = f(b) - f(a)$.
Proof:
Given $\varepsilon > 0$. There exists $\delta > 0$ such that for every partition P where,
$P = \{a = x_0, x_1, \dots \dots \dots ., x_{n-1}, x_n = b\}$.
With norm $\mu(P) - \delta\ (where\ \mu(P) = max\Delta x_i)$.
$|\sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f\ dx| < \varepsilon. [since\ t_i \in (x_{i-1}, x_i)]$.
$\Rightarrow \sum_{i=1}^{n} f(t_i)\,\Delta x_i = \int_a^b f\ \ dx$. ---------------------- (1)
By Lagrange's Mean value Theorem, $\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} = f(t_i)$.
(i.e).$\frac{f(x_i)-f(x_{i-1})}{\Delta x_i} = f(t_i)$.
$\Rightarrow f(x_i) - f(x_{i-1}) = f(t_i)\Delta x_i$. -------------------- (2)
Using (2) in (1) we get,
$\int_a^b f\ \ dx = \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]$.
$\int_a^b f\ \ dx = F(b) - F(a)$.